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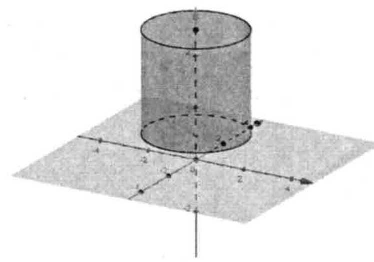
1. A certain density function is given by $f(x, y, z) = z\sqrt{x^2 + y^2}$ in kg/m^3.

   Integrate f over the cylinder $\mathcal{W}$ with $x^2 + y^2 \leq 4$ and $1 \leq z \leq 5$.

in B- coordinates

$$\mathcal{W} = \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 5 \end{cases}$$



Know:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ z = z \\ dV = r \cdot dz\, dr\, d\theta \end{cases}$$

NOTICE: $f(x,y,z) = z \cdot \sqrt{x^2+y^2} = z \cdot r$

$$\iiint_W f(x,y,z)\, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1}^{5} \underbrace{(z \cdot r)}_{f} \cdot \underbrace{r \cdot dz\, dr\, d\theta}_{dV}$$

$$= \int_0^{2\pi} \int_0^2 \left[\int_1^5 z \cdot r^3\, dz \right] dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} r^2 \right]_{z=1}^{z=5} dr\, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^2 \left(\frac{5^2 - 1^2}{2} \right) r^2\, dr \right] d\theta = \int_0^{2\pi} \left[\frac{5^2-1^2}{2} \cdot \frac{r^3}{3} \right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{5^2-1^2}{2} \right) \cdot \frac{2^3}{3}\, d\theta = \left[\left(\frac{5^2-1^2}{2} \right) \cdot \frac{2^3}{3} \cdot \theta \right]_0^{2\pi} = 2\pi \cdot \frac{2^3}{3} \cdot \left(\frac{5^2-1^2}{2} \right)$$

$$= 64 \text{ kg}.$$

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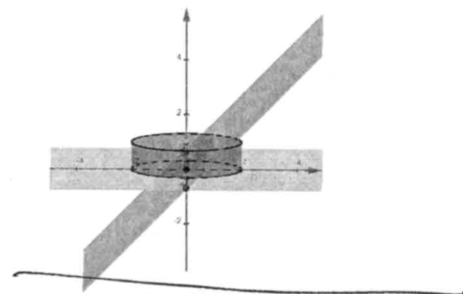
2. Set up and evaluate the integral of $f(x, y, z) = z$ on the cylinder $\mathcal{D}$ with $x^2 + y^2 \leq 4$ above the $x - y$ plane and below the plane $z = y$.

Recall

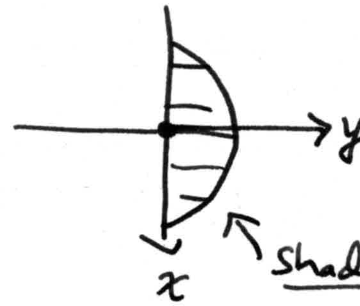
$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \\ dV = r\cdot dz\cdot dr\cdot d\theta \end{cases}$$



NOTICE

$$\mathcal{D} = \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq \pi \\ 0 \leq z \leq y = r\sin\theta \end{cases}$$

top view



compute

$$\iiint_D f\, dV = \int_{\theta=0}^{\pi}\int_{r=0}^{2}\left[\int_{z=0}^{r\cdot\sin\theta} \underbrace{z}_{f} \cdot \underbrace{r\cdot dz\, dr\, d\theta}_{dV}\right]$$

$$= \int_0^{\pi}\int_0^{2}\left[\frac{z^2 r}{2}\right]_{z=0}^{z=r\cdot\sin\theta} dr\, d\theta = \int_0^{\pi}\int_0^{2} \frac{(r\cdot\sin\theta)^2\cdot r}{2} - \frac{0}{2}\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\int_0^{2} \frac{r^3\cdot\sin^2\theta}{2}\, dr\right] d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\cdot\frac{\sin^2\theta}{2}\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi} \overbrace{\frac{2^4}{2^2\cdot 2}}^{=2} \sin^2\theta\, d\theta$$

$$\boxed{\text{Recall: } \sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$= \int_0^{\pi} \cancel{2}\, \frac{1-\cos(2\theta)}{\cancel{2}}\, d\theta = \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\theta=0}^{\theta=\pi} = \left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)$$

$$= \pi$$

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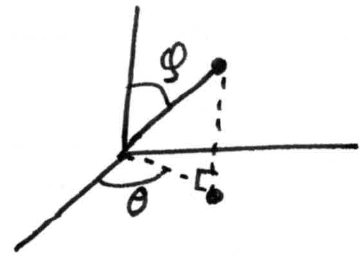
3. Use integration with spherical coordinates to compute the volume of a sphere of radius r.

Sphere $S = \begin{cases} 0 \le \rho \le r \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$



Volume of S

$$= \iiint_S 1 \cdot dV = \int_{\rho=0}^{\rho=r} \int_{\varphi=0}^{\varphi=\pi} \left[\int_{\theta=0}^{\theta=2\pi} 1 \cdot \rho^2 \cdot \sin\varphi \cdot d\theta \, d\varphi \cdot d\rho \right]$$

$$= \int_0^r \int_0^\pi \left[\rho^2 \cdot \sin\varphi \cdot \theta \right]_{\theta=0}^{\theta=2\pi} d\varphi \, d\rho$$

$$= \int_0^r \left[\int_0^\pi \rho^2 \cdot \sin\varphi \cdot 2\pi \, d\varphi \right] \cdot d\rho$$

$$= \int_0^r \left[\rho^2 \cdot (-\cos(\varphi)) \cdot 2\pi \right]_{\varphi=0}^{\varphi=\pi} d\rho$$

$$\boxed{\begin{aligned} \cos(\pi) &= (-1) \\ \cos(0) &= +1 \end{aligned}}$$

$$= \int_0^r -\rho^2 \cos(\pi) \cdot 2\pi - (-\rho\cos(0)) \, d\rho$$

$$= \int_0^r 2\rho \cdot 2\pi \, d\rho$$

$$= 2 \cdot \frac{\rho^3}{3} \cdot 2\pi \Big]_{\rho=0}^{\rho=r} = \frac{4\pi r^3}{3} - 0 = \boxed{\frac{4}{3}\pi r^3}$$

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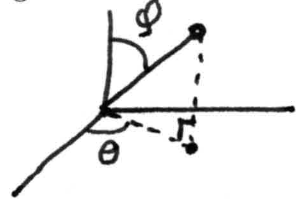
4. Suppose you buy a spherical bearing with radius $\rho = 2\ m$. The bearing's density is given by $f(x, y, z) = x^2 + y^2 + z^2$ in kg/m^3. Find the mass of the bearing.

$$S = \begin{cases} 0 \le \rho \le 2 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$



Recall

$$\rho = \sqrt{x^2+y^2+z^2}$$

so

$$f(x,y,z) = \rho^2$$

$$\iiint_S f\,dV = \int_{\rho=0}^{\rho=2}\int_{\varphi=0}^{\varphi=\pi}\left[\int_{\theta=0}^{\theta=2\pi} \underbrace{\rho^2}_{f}\cdot \underbrace{\rho^2\cdot\sin\varphi\cdot d\theta}_{dV}\right] d\varphi\, d\rho$$

$$= \int_0^2\int_0^\pi \Big[\rho^4\cdot\sin\varphi\cdot\theta\Big]_0^{2\pi} d\varphi\, d\rho = \int_0^2\left[\int_0^\pi \rho^4\cdot\sin\varphi\cdot 2\pi - 0\, d\varphi\right] d\rho$$

$$= \int_0^2 \Big[\rho^4(-\cos\varphi)\,2\pi\Big]_{\varphi=0}^{\pi} d\rho$$

$-\cos(\pi) = -(-1) = 1$

$-\cos(0) = -1$

$$= \int_0^2 \rho^4\underbrace{(-\cos(\pi))}_{1}\,2\pi - \rho\underbrace{(-\cos(0))}_{(-1)}\,2\pi\, d\rho = \int_0^2 \rho^4\cdot 2\cdot 2\pi\, d\rho$$

$$= \left[\frac{\rho^5}{5}\cdot 4\pi\right]_{\rho=0}^{\rho=2} = \frac{2^5}{5}\cdot 4\pi \text{ kg.}$$